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On Some Fundamental Integral Inequalities in Two Independent Variables

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The present paper obtains two independent variable generalizations of the integral inequalities of Gollwitzer, Langenhop, and Pachpatte. The bounds provided by these inequalities are adequate in many applications in the theory of partial differential and integral equations.

1. INTRODUCTION

The classic work "Inequalities" by Hardy *et al.* [7], which appeared in 1934, initiated the discovery of new types of inequalities and the applications of inequalities in many parts of analysis. The book "Inequalities" by Beckenbach and Bellman [1], which appeared in 1961, contains an account of some results on inequalities obtained in the period 1934–1960. An extensive survey of integral inequalities of the Gronwall type which are adequate in many applications in the theory of differential and integral equations may be found in a recent publication by Beesack [2]. A two-independent-variable generalization of Gronwall's inequality due to Wendroff given in [1, p. 154] has evoked considerable interest in recent times, as may be seen from the recent papers of Snow [11], Ghoshal and Masood [5], Young [12], Chandra and Davis [4], and Bondge and Pachpatte [3] which were motivated by certain applications in the theory of partial differential and integral equations. Our objective here is to establish two-independent-variable generalizations of the integral inequalities recently established by Gollwitzer [6], Langenhop [8], and Pachpatte [9, 10] which can be used in the analysis of various problems in the theory of partial differential and integral equations.

2. MAIN RESULTS

In this section we state and prove some interesting and useful two-independent-variable generalizations of the integral inequalities of Gollwitzer [6] and Langenhop [8].

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A useful two-independent-variable generalization of Gollwitzer's inequality given in Lemma 2 of [6, p. 642] is embodied in the following theorem.

THEOREM 1. *Let $\phi(s, t)$, $a(s, t)$, $b(s, t)$ be real-valued nonnegative continuous functions defined on $I \times I$, where $I = [0, \infty)$; let $u(s, t)$ be a positive real-valued continuous function defined on $I \times I$; and suppose further that the inequality*

$$u(s, t) \geqslant \phi(x, y) - a(s, t) \left(\int_x^s \int_y^t b(m, n)\, \phi(m, n)\, dm\, dn \right) \tag{1}$$

is satisfied for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$. Then

$$u(s, t) \geqslant \phi(x, y) \exp \left(-a(s, t) \left(\int_x^s \int_y^t b(m, n)\, dm\, dn \right) \right), \tag{2}$$

for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$.

Proof. Rewrite (1) as

$$\phi(x, y) \leqslant u(s, t) + a(s, t) \left(\int_x^s \int_y^t b(m, n)\, \phi(m, n)\, dm\, dn \right). \tag{3}$$

For fixed s and t in the interval I we define for $0 \leqslant x \leqslant s$, $0 \leqslant y \leqslant t$,

$$\begin{aligned} r(x, y) &= u(s, t) + a(s, t) \left(\int_x^s \int_y^t b(m, n)\, \phi(m, n)\, dm\, dn \right), \\ r(x, t) &= r(s, y) = u(s, t); \end{aligned} \tag{4}$$

then from (4) we have

$$r_{xy}(x, y) = a(s, t)\, b(x, y)\, \phi(x, y),$$

which in view of (3) implies

$$r_{xy}(x, y) \leqslant a(s, t)\, b(x, y)\, r(x, y),$$

i.e.,

$$\frac{r_{xy}(x, y)}{r(x, y)} \leqslant a(s, t)\, b(x, y). \tag{5}$$

From (5) we observe that

$$\frac{r(x, y)\, r_{xy}(x, y)}{r^2(x, y)} \leqslant a(s, t)\, b(x, y) + \frac{r_x(x, y)\, r_y(x, y)}{r^2(x, y)},$$

i.e.,

$$\frac{\partial}{\partial y}\left(\frac{r_x(x, y)}{r(x, y)}\right) \leqslant a(s, t)\, b(x, y).$$

Now integrating both sides of the above inequality with respect to y from y to t we have

$$\frac{r_x(x, t)}{r(x, t)} - \frac{r_x(x, y)}{r(x, y)} \leqslant a(s, t) \int_y^t b(x, n)\, dn.$$

Integrating both sides of the above inequality with respect to x from x to s we have

$$r(x, y) \leqslant u(s, t) \exp\left(a(s, t)\left(\int_x^s \int_y^t b(m, n)\, dm\, dn\right)\right). \tag{6}$$

The desired bound in (2) follows from (3) and (6) since s and t are arbitrary in the interval I.

As an application of Theorem 1 we next establish the following two-independent-variable generalization of the Gollwitzer's inequality given in [6, Theorem 1] for the lower bound on an unknown function.

THEOREM 2. *Let $\phi(s, t)$, $a(s, t)$, $b(s, t)$, and $u(s, t)$ be as defined in Theorem 1; let $H(r)$ be a positive, continuous, strictly increasing, convex, and submultiplicative function for $r > 0$, $H(0) = 0$, $\lim_{r\to\infty} H(r) = \infty$; let $\alpha(s, t)$, $\beta(s, t)$ be positive continuous functions defined on $I \times I$, and $\alpha(s, t) + \beta(s, t) = 1$. Suppose further that the inequality*

$$u(s, t) \geqslant \phi(x, y) - a(s, t)\, H^{-1}\left(\int_x^s \int_y^t b(m, n)\, H(\phi(m, n))\, dm\, dn\right) \tag{7}$$

is satisfied for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$. Then

$$u(s, t) \geqslant \alpha(s, t)\, H^{-1}\Big(\alpha^{-1}(s, t)\, H(\phi(x, y))$$
$$\times \exp\left(-\beta(s, t)\, H(a(s, t)\, \beta^{-1}(s, t)) \int_x^s \int_y^t b(m, n)\, dm\, dn\right)\Big), \tag{8}$$

for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$.

Proof. Rewrite (7) as

$$\phi(x, y) \leqslant \alpha(s, t)\, u(s, t)\, \alpha^{-1}(s, t) + \beta(s, t)\, a(s, t)\, \beta^{-1}(s, t)$$
$$\times H^{-1}\left(\int_x^s \int_y^t b(m, n)\, H(\phi(m, n))\, dm\, dn\right).$$

Since H is convex, submultiplicative, and monotonic we have

$$\alpha(s,t)\,H(u(s,t)\,\alpha^{-1}(s,t)) \geqslant H(\phi(x,y)) - \beta(s,t)\,H(a(s,t)\,\beta^{-1}(s,t)) \times \left(\int_x^s\int_y^t b(m,n)\,H(\phi(m,n))\,dm\,dn\right).$$

Now an application of Theorem 1 yields the desired bound in (8).

We next establish the following two-independent-variable generalization of the integral inequality established by Langenhop [8].

THEOREM 3. *Let $u(s,t)$, $a(s,t)$, and $b(s,t)$ be as defined in Theorem 1; let $W(r)$ be a positive, continuous, monotonic, nondecreasing function for $r > 0$, $W(0) = 0$, and $(\partial/\partial y)W(r(x,y)) = W_y(r(x,y)) \geqslant 0$; and suppose further that the inequality*

$$u(s,t) \geqslant u(x,y) - a(s,t)\left(\int_x^s\int_y^t b(m,n)\,W(u(m,n))\,dm\,dn\right) \tag{9}$$

is satisfied for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$. Then for $s_1, t_1 \in I$, $0 \leqslant x \leqslant s \leqslant s_1$, $0 \leqslant y \leqslant t \leqslant t_1$,

$$u(s,t) \geqslant \Omega^{-1}\left[\Omega(u(x,y)) - a(s,t)\left(\int_x^s\int_y^t b(m,n)\,dm\,dn\right)\right], \tag{10}$$

where

$$\Omega(r) = \int_{r_0}^{r}\frac{ds}{W(s)}, \qquad r \geqslant r_0 > 0, \tag{11}$$

Ω^{-1} is the inverse function of Ω, and

$$\Omega(u(x,y)) - a(s,t)\left(\int_x^s\int_y^t b(m,n)\,dm\,dn\right) \in \mathrm{Dom}(\Omega^{-1}),$$

for $0 \leqslant x \leqslant s \leqslant s_1$, $0 \leqslant y \leqslant t \leqslant t_1$.

Proof. Rewrite (9) as

$$u(x,y) \leqslant u(s,t) + a(s,t)\left(\int_x^s\int_y^t b(m,n)\,W(u(m,n))\,dm\,dn\right). \tag{12}$$

For fixed s and t in the interval I we define for $0 \leqslant x \leqslant s$, $0 \leqslant y \leqslant t$,

$$\begin{aligned} r(x,y) &= u(s,t) + a(s,t)\left(\int_x^s\int_y^t b(m,n)\,W(u(m,n))\,dm\,dn\right),\\ r(x,t) &= r(s,y) = u(s,t); \end{aligned} \tag{13}$$

then from (13) we have

$$r_{xy}(x, y) = a(s, t)\, b(x, y)\, W(u(x, y)),$$

which in view of (12) implies

$$r_{xy}(x, y) \leqslant a(s, t)\, b(x, y)\, W(r(x, y)),$$

i.e.,

$$\frac{r_{xy}(x, y)}{W(r(x, y))} \leqslant a(s, t)\, b(x, y). \tag{14}$$

From (14) we observe that

$$\frac{W(r(x, y))\, r_{xy}(x, y)}{W^2(r(x, y))} \leqslant a(s, t)\, b(x, y) + \frac{W_y(r(x, y))\, r_x(x, y)}{W^2(r(x, y))},$$

i.e.,

$$\frac{\partial}{\partial y}\left(\frac{r_x(x, y)}{W(r(x, y))}\right) \leqslant a(s, t)\, b(x, y).$$

Now integrating both sides of the above inequality with respect to y from y to t we have

$$\frac{r_x(x, t)}{W(r(x, t))} - \frac{r_x(x, y)}{W(r(x, y))} \leqslant a(s, t) \int_y^t b(x, n)\, dn. \tag{15}$$

From (11) and (15) we observe that

$$\Omega_x(r(x, t)) - \Omega_x(r(x, y)) \leqslant a(s, t) \int_y^t b(x, n)\, dn.$$

Integrating both sides of the above inequality with respect to x from x to s we have

$$\Omega(r(x, y)) \leqslant \Omega(u(s, t)) + a(s, t) \left(\int_x^s \int_y^t b(m, n)\, dm\, dn\right),$$

which implies

$$\Omega(u(s, t)) \geqslant \Omega(u(x, y)) - a(s, t) \left(\int_x^s \int_y^t b(m, n)\, dm\, dn\right). \tag{16}$$

The desired bound in (10) follows from (16). The intervals of real numbers s and t are obvious.

3. Further Inequalities

In this section we establish two-independent-variable generalizations of the integral inequalities recently established by Pachpatte [9, 10] which can be used in some applications in the theory of Hyperbolic partial integral and integrodifferential equations.

Our first result deals with the two-independent-variable generalization of the integral inequality recently established by Pachpatte in [10, Theorem 1].

THEOREM 4. *Let $\phi(s, t)$, $a(s, t)$, $b(s, t)$, and $c(s, t)$ be real-valued nonnegative continuous functions defined on $I \times I$; let $u(s, t)$ be a positive real-valued continuous function defined on $I \times I$; and suppose further that the inequality*

$$\begin{aligned} u(s, t) \geqslant \phi(x, y) - a(s, t) \Big[\int_x^s \int_y^t b(m, n)\, \phi(m, n)\, dm\, dn \\ + \int_x^s \int_y^t b(m, n) \Big(\int_m^s \int_n^t c(\xi, \zeta)\, \phi(\xi, \zeta)\, d\xi\, d\zeta \Big) dm\, dn \Big], \end{aligned} \tag{17}$$

is satisfied for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$. Then

$$\begin{aligned} u(s, t) \geqslant \phi(x, y) \Big[1 + a(s, t) \Big(\int_x^s \int_y^t b(m, n) \\ \times \exp \Big(\int_m^s \int_n^t [a(s, t)\, b(\xi, \zeta) + c(\xi, \zeta)]\, d\xi\, d\zeta \Big) dm\, dn \Big) \Big]^{-1}, \end{aligned} \tag{18}$$

for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$.

Proof. Rewrite (17) as

$$\begin{aligned} \phi(x, y) \leqslant u(s, t) + a(s, t) \Big[\int_x^s \int_y^t b(m, n)\, \phi(m, n)\, dm\, dn \\ + \int_x^s \int_y^t b(m, n) \Big(\int_m^s \int_n^t c(\xi, \zeta)\, \phi(\xi, \zeta)\, d\xi\, d\zeta \Big) dm\, dn \Big]. \end{aligned} \tag{19}$$

For fixed s and t in the interval I, we define for $0 \leqslant x \leqslant s$, $0 \leqslant y \leqslant t$,

$$\begin{aligned} r(x, y) = u(s, t) + a(s, t) \Big[\int_x^s \int_y^t b(m, n)\, \phi(m, n)\, dm\, dn \\ + \int_x^s \int_y^t b(m, n) \Big(\int_m^s \int_n^t c(\xi, \zeta)\, \phi(\xi, \zeta)\, d\xi\, d\zeta \Big) dm\, dn \Big], \end{aligned} \tag{20}$$

$$r(x, t) = r(s, y) = u(s, t);$$

then from (20) we have

$$r_{xy}(x, y) = a(s, t)\, b(x, y) \left[\phi(x, y) + \int_x^s \int_y^t c(\xi, \zeta)\, \phi(\xi, \zeta)\, d\xi\, d\zeta\right],$$

which in view of (19) implies

$$r_{xy}(x, y) \leqslant a(s, t)\, b(x, y) \left[r(x, y) + \int_x^s \int_y^t c(\xi, \zeta)\, r(\xi, \zeta)\, d\xi\, d\zeta\right]. \tag{21}$$

Define

$$\begin{aligned} v(x, y) &= r(x, y) + \int_x^s \int_y^t c(\xi, \zeta)\, r(\xi, \zeta)\, d\xi\, d\zeta, \\ v(s, y) &= v(x, t) = u(s, t); \end{aligned} \tag{22}$$

then from (22) we have

$$v_{xy}(x, y) = r_{xy}(x, y) + c(x, y)\, r(x, y),$$

which in view of (21) and the fact that $r(x, y) \leqslant v(x, y)$ from (22) implies

$$v_{xy}(x, y) \leqslant [a(s, t)\, b(x, y) + c(x, y)]\, v(x, y)$$

which by following an argument similar to that in the proof of Theorem 1 yields the estimate for $v(x, y)$ such that

$$v(x, y) \leqslant u(s, t) \exp\left(\int_x^s \int_y^t [a(s, t)\, b(\xi, \zeta) + c(\xi, \zeta)]\, d\xi\, d\zeta\right).$$

Substituting this bound on $v(x, y)$ in (21) we have

$$r_{xy}(x, y) \leqslant a(s, t)\, b(x, y)\, u(s, t) \exp\left(\int_x^s \int_y^t [a(s, t)\, b(\xi, \zeta) + c(\xi, \zeta)]\, d\xi\, d\zeta\right).$$

Now integrating both sides of the above inequality with respect to y from y to t we have

$$\begin{aligned} &r_x(x, t) - r_x(x, y) \\ &\qquad \leqslant a(s, t)\, u(s, t) \int_y^t b(x, n) \exp\left(\int_x^s \int_n^t [a(s, t)\, b(\xi, \zeta) + c(\xi, \zeta)]\, d\xi\, d\zeta\right) dn. \end{aligned}$$

Integrating both sides of the above inequality with respect to x from x to s we have

$$r(x, y) \leqslant u(s, t) \Big[1 + a(s, t) \Big(\int_x^s \int_y^t b(m, n) \times \exp\Big(\int_m^s \int_n^t [a(s, t)\, b(\xi, \zeta) + c(\xi, \zeta)]\, d\xi\, d\zeta\Big)\, dm\, dn\Big)\Big]. \tag{23}$$

The desired bound in (18) follows from (19) and (23) since s and t are arbitrary in the interval I.

We now apply Theorem 4 to establish the following two-independent-variable generalization of the integral inequality recently established by Pachpatte [10, Theorem 2].

THEOREM 5. *Let $\phi(s, t)$, $a(s, t)$, $b(s, t)$, $c(s, t)$, and $u(s, t)$ be as defined in Theorem 4; let $H(r)$, $\alpha(s, t)$ and $\beta(s, t)$ be as defined in Theorem 2; and suppose further that the inequality*

$$u(s, t) \geqslant \phi(x, y) - a(s, t)\, H^{-1}\Big[\int_x^s \int_y^t b(m, n)\, H(\phi(m, n))\, dm\, dn + \int_x^s \int_y^t b(m, n) \Big(\int_m^s \int_n^t c(\xi, \zeta)\, H(\phi(\xi, \zeta))\, d\xi\, d\zeta\Big)\, dm\, dn\Big] \tag{24}$$

is satisfied for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$, then

$$u(s, t) \geqslant \alpha(s, t)\, H^{-1}\Big[\alpha^{-1}(s, t)\, H(\phi(x, y)) \Big\{1 + \beta(s, t)\, H(a(s, t)\, \beta^{-1}(s, t)) \times \int_x^s \int_y^t b(m, n) \exp\Big(\int_m^s \int_n^t [\beta(s, t)\, H(a(s, t)\, \beta^{-1}(s, t))\, b(\xi, \zeta) + c(\xi, \zeta)] \times d\xi\, d\zeta\Big)\, dm\, dn\Big\}^{-1}\Big], \tag{25}$$

for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$.

Proof. Rewrite (24) as

$$\phi(x, y) \leqslant \alpha(s, t)\, u(s, t)\, \alpha^{-1}(s, t) + \beta(s, t)\, a(s, t)\, \beta^{-1}(s, t)\, H^{-1}\Big[\int_x^s \int_y^t b(m, n)\, H(\phi(m, n))\, dm\, dn + \int_x^s \int_y^t b(m, n) \Big(\int_m^s \int_n^t c(\xi, \zeta)\, H(\phi(\xi, \zeta))\, d\xi\, d\zeta\Big)\, dm\, dn\Big].$$

Since H is convex, submultiplicative, and monotonic we have

$$\alpha(s, t)\, H(u(s, t)\, \alpha^{-1}(s, t))$$
$$\geqslant H(\phi(x, y)) - \beta(s, t)\, H(a(s, t)\, \beta^{-1}(s, t)) \left[\int_x^s \int_y^t b(m, n)\, H(\phi(m, n))\, dm\, dn\right].$$

Now an application of Theorem 4 yields the desired bound in (25).

To this end we present the following two-independent-variable generalization of the integral inequality established by Pachpatte in [9, Theorem 3].

THEOREM 6. *Let $u(s, t)$, $a(s, t)$, $b(s, t)$, and $c(s, t)$ be as defined in Theorem 4; let $G(r)$ be a positive, continuous, strictly increasing, subadditive and submultiplicative function for $r > 0$, $H(0) = 0$; let G^{-1} denote the inverse function of G; and suppose further that the inequality*

$$\begin{aligned} u(s, t) \geqslant u(x, y) - a(s, t)\, G^{-1} \Big[& \int_x^s \int_y^t b(m, n)\, G(u(m, n))\, dm\, dn \\ & + \int_x^s \int_y^t b(m, n) \left(\int_m^s \int_n^t c(\xi, \zeta)\, G(u(\xi, \zeta))\, d\xi\, d\zeta\right) dm\, dn \Big] \end{aligned} \tag{26}$$

is satisfied for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$; then

$$\begin{aligned} u(s, t) \geqslant u(x, y)\, G^{-1} \Big\{ \Big[1 + G(a(s, t)) & \int_x^s \int_y^t b(m, n) \\ & \times \exp\left(\int_m^s \int_n^t [b(\xi, \zeta)\, G(a(s, t)) + c(\xi, \zeta)]\, d\xi\, d\zeta\right) dm\, dn \Big]^{-1} \Big\}, \end{aligned} \tag{27}$$

for $0 \leqslant x \leqslant s < \infty$, $0 \leqslant y \leqslant t < \infty$.

Proof. Rewrite (26) as

$$\begin{aligned} u(x, y) \leqslant u(s, t) + a(s, t)\, G^{-1} \Big[& \int_x^s \int_y^t b(m, n)\, G(u(m, n))\, dm\, dn \\ & + \int_x^s \int_y^t b(m, n) \left(\int_m^s \int_n^t c(\xi, \zeta)\, G(u(\xi, \zeta))\, d\xi\, d\zeta\right) dm\, dn \Big]. \end{aligned} \tag{28}$$

Since G is subadditive and submultiplicative we have from (28)

$$\begin{aligned} G(u(x, y)) \leqslant G(u(s, t)) + G(a(s, t)) \Big[& \int_x^s \int_y^t b(m, n)\, G(u(m, n))\, dm\, dn \\ & + \int_x^s \int_y^t b(m, n) \left(\int_m^s \int_n^t c(\xi, \zeta)\, G(u(\xi, \zeta))\, d\xi\, d\zeta\right) dm\, dn \Big]. \end{aligned} \tag{29}$$

Defining $r(x, y)$ by the right member of (29) and by following an argument similar to that in the proof of Theorem 4, with suitable modifications, we obtain the desired bound in (27).

4. Some Applications

In this section we present some applications of the inequalities established in this paper to obtain the lower bounds on the solutions of a class of hyperbolic partial differential and integrodifferential equations. Each of these applications could be stated formally as a theorem. This has not been done so as not to obscure the essential ideas with technical details.

Example 1. As a first application we obtain the lower bound on the solution of a nonlinear hyperbolic partial differential equation of the form

$$u_{xy}(x, y) = F[x, y, u(r, y)], \tag{30}$$

with the given boundary conditions $u(x, t) = u(s, y) = u(s, t)$, where the functions u and f are real valued, defined, and continuous on the respective domains of their definitions and

$$|F[x, y, u(x, y)]| \leqslant b(x, y)\, W(|u(x, y)|), \tag{31}$$

where b and W are as defined in Theorem 3. Integrating (30) first with respect to y from y to t and then with respect to x from x to s we have

$$u(x, y) = u(s, t) + \int_x^s \int_y^t F[m, n, u(m, n)]\, dm\, dn. \tag{32}$$

Using (31) in (32) we have

$$|u(x, y)| \leqslant |u(s, t)| + \int_x^s \int_y^t b(m, n)\, W(|u(m, n)|)\, dm\, dn,$$

i.e.,

$$|u(s, t)| \geqslant |u(x, y)| - \int_x^s \int_y^t b(m, n)\, W(|u(m, n)|)\, dm\, dn.$$

Now a suitable application of Theorem 3 yields

$$|u(s, t)| \geqslant \Omega^{-1}\left[\Omega(|u(x, y)|) - \int_x^s \int_y^t b(m, n)\, dm\, dn\right], \tag{33}$$

where Ω and Ω^{-1} are as defined in Theorem 3. Thus the right-hand side in (33) gives us the lower bound on the solution $u(s, t)$ of (30).

EXAMPLE 2. As a second application, we establish the lower bound on the solution of a nonlinear hyperbolic partial integrodifferential equation of the form

$$u_{xy}(x, y) = F\left[x, y, u(x, y), \int_x^s \int_y^t k(x, y, m, n, u(m, n))\, dm\, dn\right], \tag{34}$$

with the given boundary conditions $u(x, t) = u(s, y) = u(s, t)$, where u, k, and F are real-valued continuous functions defined on the respective domains of their definitions and the functions k and F involved in (34) satisfy

$$| k(x, y, m, n, u(m, n))| \leqslant c(m, n)\, | u(m, n)|, \tag{35}$$

$$| F[x, y, u(x, y), v]| \leqslant b(x, y)[| u(x, y)| + | v |], \tag{36}$$

where b and c are as defined in Theorem 4. Integrating (34) as in example 1 we have

$$u(x, y) = u(s, t) + \int_x^s \int_y^t F\left[m, n, u(m, n), \int_m^s \int_n^t k(m, n, \xi, \zeta, u(\xi, \zeta))\, d\xi\, d\zeta\right] dm\, dn. \tag{37}$$

Using (35) and (36) in (37) we have

$$| u(x, y)| \leqslant | u(s, t)| + \int_x^s \int_y^t b(m, n)\, | u(m, n)|\, dm\, dn$$
$$+ \int_x^s \int_y^t b(m, n) \left[\int_m^s \int_n^t c(\xi, \zeta)\, | u(\xi, \zeta)|\, d\xi\, d\zeta\right] dm\, dn,$$

i.e.,

$$| u(s, t)| \geqslant | u(x, y)| - \left[\int_x^s \int_y^t b(m, n)\, | u(m, n)|\, dm\, dn\right.$$
$$\left. + \int_x^s \int_y^t b(m, n) \left(\int_m^s \int_n^t c(\xi, \zeta)\, | u(\xi, \zeta)|\, d\xi\, d\zeta\right) dm\, dn\right].$$

Now a suitable application of Theorem 4 yields

$$| u(s, t)| \geqslant | u(x, y)| \left[1 + \int_x^s \int_y^t b(m, n)\right.$$
$$\left. \times \exp\left(\int_m^s \int_n^t [b(\xi, \zeta) + c(\xi, \zeta)]\, d\xi\, d\zeta\right) dm\, dn\right]^{-1},$$

which gives us the lower bound on the solution $u(s, t)$ of (34).

REFERENCES

1. E. F. BECKENBACH AND R. BELLMAN, "Inequalities," Springer-Verlag, Berlin, 1961.
2. P. R. BEESACK, "Gronwall Inequalities," Carleton Mathematical Lecture Notes No. 11, May, 1975.
3. B. K. BONDGE AND B. G. PACHPATTE, On nonlinear integral inequalities of the Wendroff type, *J. Math. Anal. Appl.* **70** (1979), 161–169.
4. J. CHANDRA AND P. W. DAVIS, Linear generalization of Gronwall's inequality, *Proc. Amer. Math. Soc.* **60** (1976), 157–160.
5. S. K. GHOSHAL AND M. A. MASOOD, Gronwall's vector inequality and its application to a class of non-self-adjoint linear and nonlinear hyperbolic partial differential equations, *J. Indian Math. Soc.* **38** (1974), 383–394.
6. H. E. GOLLWITZER, A note on a functional inequality, *Proc. Amer. Math. Soc.* **23** (1969), 642–647.
7. G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, "Inequalities," Second ed., Cambridge Univ. Press, London/New York, 1952.
8. C. E. LANGENHOP, Bounds on the norm of a solution of a general differential equation, *Proc. Amer. Math. Soc.* **11** (1960), 795–799.
9. B. G. PACHPATTE, A note on integral inequalities of the Bellman–Bihari type, *J. Math. Anal. Appl.* **49** (1975), 295–301.
10. B. G. PACHPATTE, A note on an integral inequality, *J. Math. Phys. Sci.* **9** (1975), 11–14.
11. D. R. SNOW, Gronwall's inequality for systems of partial differential equations in two independent variables, *Proc. Amer. Math. Soc.* **33** (1972), 46–54.
12. E. C. YOUNG, Gronwall's inequality in n independent variables, *Proc. Amer. Math. Soc.* **41** (1973), 241–244.